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A note on a cycle partition problem ☆

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ABSTRACT

Let G be any graph, and let $c(G)$ denote the circumference of G. If, for every pair c_1, c_2 of positive integers satisfying $c_1 + c_2 = c(G)$, the vertex set of G admits a partition into two sets V_1 and V_2 such that V_i induces a graph of circumference at most c_i, $i = 1, 2$, then G is said to be c-partitionable. In [M.H. Nielsen, On a cycle partition problem, Discrete Math. 308 (2008) 6339–6347], it is conjectured that every graph is c-partitionable. In this paper, we verify this conjecture for a graph with a longest cycle that is a dominating cycle. Moreover, we prove that G is c-partitionable if $c(G) \geq |V(G)| - 3$.

1. Introduction

Throughout this paper, we consider only finite undirected graphs without loops and multiple edges. For notation and terminology not defined here, we refer to West [1].

The vertex set and edge set of a graph G are denoted by $V = V(G)$ and $E = E(G)$, respectively. $\alpha(G)$ and $\kappa(G)$ denote the independence number and the connectivity of G, respectively. If $k \leq \alpha(G)$, we define $\sigma_k(G)$ as the minimum value of the degree sum of any k independent vertices of G; otherwise, we set $\sigma_k(G) = +\infty$. For $S \subseteq V$, let $\langle S \rangle$ denote the subgraph induced by S. For subgraphs H and K of G, let $G - H := \langle V \setminus V(H) \rangle$, and let $N_H(K)$ denote the set of vertices in H that are adjacent to some vertex in K. Moreover, we let $N(K) := N_{G-K}(K)$. In particular, if K consists of one vertex v, we omit the brackets, and we use $d_H(v) = |N_H(v)|$ and $d(v) = |N(v)|$. A cycle C is called a *dominating cycle* if $G - C$ is edgeless.

The *circumference* $c(G)$ of G is defined as follows. If G is edgeless, then $c(G) = 1$; if G is acyclic but contains an edge, then $c(G) = 2$; finally, if G contains a cycle, then $c(G)$ is the length of a longest cycle in G. A longest path in a graph G is called a *detour* of G. The number of vertices in a detour of G is called the *detour order* of G, and is denoted by $\tau(G)$.

A graph G is said to be *τ-partitionable* if, for every pair (a, b) of positive integers with $a + b = \tau(G)$, V has a partition $V = V_1 \cup V_2$ such that $\tau(\langle V_1 \rangle) \leq a$ and $\tau(\langle V_2 \rangle) \leq b$.

The following conjecture, known as the Path Partition Conjecture (PPC), was posed by Laborde et al. [2] in 1982.

Conjecture 1. *Every graph is τ-partitionable.*

In recent years, a number of results have been reported in support of the conjecture (see, e.g., [3–7]). However, the general conjecture seems to be quite difficult to settle. Bondy [8] considered the directed version of the conjecture, and some papers have dealt with that conjecture (see, e.g., [9,10]).

Notice that similar concepts have also been defined for other parameters. A graph G is called *Δ-partitionable* (where Δ stands for the maximum degree of G) if, for every pair (a, b) of positive integers with $a+b = \Delta-1$, there is a partition (V_1, V_2)

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of V such that $\Delta(\langle V_1\rangle) \leq a$ and $\Delta(\langle V_2\rangle) \leq b$. In [11], Lovász proved that every graph is Δ-partitionable. Stiebitz proved a dual-type result with respect to the minimum degree [12]. Recently, Nielsen [13] defined a cycle analog of the path partition problem in a natural way. If c_1 and c_2 are positive integers and (V_1, V_2) is a partition of $V(G)$ such that $c(\langle V_i\rangle) \leq c_i,\ i = 1, 2$, then we say that (V_1, V_2) is a *(c_1, c_2)-partition* of G and that G is *(c_1, c_2)-partitionable*. If G is (c_1, c_2)-partitionable for every pair (c_1, c_2) of positive integers with $c_1 + c_2 = c(G)$, then G is said to be *c-partitionable*. For more details on the PPC and related problems, we refer the reader to [14,3,13] and the references therein.

The following Conjecture 2 was formulated by Nielsen [13].

Conjecture 2 ([13]). *Every graph is c-partitionable.*

Among other results, the following Theorem 1.1 and Proposition 1.2 have been proved in [13].

Theorem 1.1 ([13]). *A graph G is c-partitionable if G belongs to one of the following graph classes.*
(a) *Planar graphs.* (b) *Bipartite graphs.* (c) *Comparability graphs.* (d) *Chordal graphs.* (e) *Claw-free graphs.* (f) *Complete multipartite graphs.*

Proposition 1.2 ([13]). *Every graph G of circumference at least $|V(G)| - 1$ is c-partitionable.*

The following result of Nielsen shows that a graph with circumference at most 9 is also c-partitionable.

Theorem 1.3 ([13]). *Let G be a graph, and let $1 \leq c_1 \leq c_2$ be positive integers such that $c_1 + c_2 = c(G)$ and $c_1 \leq 4$. Then G is (c_1, c_2)-partitionable.*

In this paper, we prove that a graph is c-partitionable if it admits a longest cycle which is a dominating cycle.

Theorem 1.4. *If a graph G admits a longest cycle that is a dominating cycle, then G is c-partitionable.*

Moreover, we extend Proposition 1.2 by decreasing the upper bound on the circumference of a c-partitionable graph to $|V(G)|-3$. Note that Frick and Schiermeyer proved that a graph G with detour order at least $|V(G)|-3$ is τ-partitionable [6].

Proposition 1.5. *Every graph G of circumference at least $|V(G)| - 3$ is c-partitionable.*

There are some Dirac-type and Ore-type conditions for a graph to admit a dominating longest cycle (see [15–22]). By Theorem 1.4, all these are the conditions for a graph to be c-partitionable, and hence we have the following two corollaries.

Corollary 1.6. *Let G be a 2-connected graph of order n. G is c-partitionable if one of the following holds.*
(a) $\delta \geq \frac{n-3}{2}$. (b) $\delta \geq \frac{n+2}{3}$. (c) $\sigma_2(G) \geq n - 3$. (d) $\sigma_3(G) \geq n + 2$.

Corollary 1.7. *Let G be a 3-connected graph of order n. G is c-partitionable if one of the following holds.*
(a) $\sigma_4 \geq n + 2\kappa$. (b) $n \geq 13,\ \sigma_4(G) \geq \frac{4}{3}n + \frac{5}{3}$. (c) $\sigma_4(G) \geq \frac{3}{2}n + 1$. (d) $\sigma_4(G) \geq n + \kappa + 3$.

2. Proof of the main results

Proof of Theorem 1.4. If $c(G) \leq 2$, then G is bipartite, and, by Theorem 1.1, G is c-partitionable. So we may assume that $c(G) \geq 3$ and that G has a longest cycle that is a dominating cycle. Then, obviously, it suffices to consider the case when G is connected.

Suppose that G is a connected graph and that C is a longest cycle (with a given cyclic orientation) that is a dominating cycle. Then $G - C = \{v_1, v_2, \ldots, v_k\}$ is an independent set. Suppose that $1 \leq c_1 \leq c_2$ are two integers with $c_1 + c_2 = c(G)$. For distinct vertices x, y on C, we use $C(x, y)$ to denote the set of vertices on C from x to y (not including x and y) with respect to the orientation. From the maximality of C, it is not difficult to get the following observations.

Observation. (a) If $x, y \in N(v_i)$ with $x \neq y$, then $|C(x, y)| \geq 1$.
(b) $|N(v_i)| \leq \left\lfloor \frac{c(G)}{2} \right\rfloor,\ i = 1, 2, \ldots, k$.
(c) Let $x_1, x_2 \in N(v_i), x'_1, x'_2 \in N(v_j),\ i \neq j$, and let these four distinct vertices appear on C in the following order: x_1, x'_1, x_2, x'_2. Then $|C(x_1, x'_1) \cup C(x_2, x'_2)| \geq 2$ and $|C(x'_1, x_2) \cup C(x'_2, x_1)| \geq 2$.

Case 1. $\left|\bigcup_{i=1}^{k} N(v_i)\right| \leq c_2$.

In this case, since $\left|\bigcup_{i=1}^{k} N(v_i)\right| \leq c_2$, we let V_2 be a set of order c_2 satisfying $\bigcup_{i=1}^{k} N(v_i) \subseteq V_2 \subseteq C$, and let $V_1 = V(G) \setminus V_2$.

Obviously, $c(\langle V_2\rangle) \leq |V_2| = c_2$. Since every vertex of $G - C$ is of degree zero in $\langle V_1\rangle$, v_i is not on any cycle in $\langle V_1\rangle$, for $i = 1, 2, \ldots, k$. Hence, we have $c(\langle V_1\rangle) \leq |V_1 - \{v_1, \ldots, v_k\}| = c_1$. Thus (V_1, V_2) is a (c_1, c_2)-partition of G.

Case 2. $\left|\bigcup_{i=1}^{k} N(v_i)\right| > c_2$.

Claim 2.1. $c(G) \geq 2\left|\bigcup_{i=1}^{k} N(v_i)\right| - k.$

Proof of Claim 2.1. First, suppose that $N(v_i) \bigcap N(v_j) = \emptyset$ for $i \neq j$. By observation, it is easy to see that

$$\begin{aligned} c(G) &\geq |N(v_1)| + |N(v_2)| + \cdots + |N(v_k)| + (|N(v_1)| - 1) + \cdots + (|N(v_k)| - 1) \\ &= 2\left|\bigcup_{i=1}^{k} N(v_i)\right| - k. \end{aligned}$$

Next, suppose that there are distinct vertices $v_i, v_j \in G - C$ such that $N(v_i) \bigcap N(v_j) \neq \emptyset$. Set $N'(v_1) = N(v_1)$, $N'(v_2) = N(v_2) \setminus N(v_1), \ldots, N'(v_k) = N(v_k) \setminus \bigcup_{i=1}^{k-1} N(v_i)$. Then, again by observation, we have

$$\begin{aligned} c(G) &\geq |N'(v_1)| + |N'(v_2)| + \cdots + |N'(v_k)| + (|N'(v_1)| - 1) + \cdots + (|N'(v_k)| - 1) \\ &= 2(|N'(v_1)| + |N'(v_2)| + \cdots + |N'(v_k)|) - k \\ &= 2\left|\bigcup_{i=1}^{k} N(v_i)\right| - k. \quad \square \end{aligned}$$

Claim 2.2. $\left|\bigcup_{i=1}^{k} N(v_i)\right| \leq \left\lceil \frac{c(G)}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor.$

Proof of Claim 2.2. From Claim 2.1, we know that $\left|\bigcup_{i=1}^{k} N(v_i)\right| \leq \frac{c(G)+k}{2}$. Since $\left|\bigcup_{i=1}^{k} N(v_i)\right|$ is an integer, we have $\left|\bigcup_{i=1}^{k} N(v_i)\right| \leq \left\lfloor \frac{c(G)+k}{2} \right\rfloor \leq \left\lceil \frac{c(G)}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor$. $\square$

Since $\left\lceil \frac{c(G)}{2} \right\rceil \leq c_2$, by Claim 2.2, we have $\left|\bigcup_{i=1}^{k} N(v_i)\right| \leq c_2 + \left\lfloor \frac{k}{2} \right\rfloor$. Without loss of generality, we may assume that $d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_k)$. We choose a vertex $u_1 \in N(v_1)$ such that $d_{G-C}(u_1) = \max\{d_{G-C}(w) | w \in N(v_1)\}$, and let $u_1 \in V_2$. Now, consider the graph $G_1 = G - u_1$. In G_1, we relabel $v_1, v_2, \ldots, v_k$ as $v_1', v_2', \ldots, v_k'$ such that $d_{G_1}(v_1') \geq d_{G_1}(v_2') \geq \cdots \geq d_{G_1}(v_k')$. Then, we choose $u_2 \in N_{G_1}(v_1')$ such that $d_{G_1-C}(u_2) = \max\{d_{G_1-C}(w) | w \in N_{G_1}(v_1')\}$ and let $u_2 \in V_2$. Continue this process until $|V_2| = c_2$. Note this is possible because $\left|\bigcup_{i=1}^{k} N(v_i)\right| > c_2$. Obviously, $V_2 \subseteq \bigcup_{i=1}^{k} N(v_i)$. Further, since $c_2 < \left|\bigcup_{i=1}^{k} N(v_i)\right| \leq c_2 + \left\lfloor \frac{k}{2} \right\rfloor$, we know that $\left|\bigcup_{i=1}^{k} N(v_i) - V_2\right| \leq \left\lfloor \frac{k}{2} \right\rfloor$. Let $V_1 = V \setminus V_2$. Then, by Claim 2.2 and the choice of V_2, it is not difficult to see that each v_i has degree at most 1 in $\langle V_1 \rangle$, and consequently $c(\langle V_1 \rangle) \leq |V_1 - \{v_1, v_2, \ldots, v_k\}| = c_1$. So G is (c_1, c_2)-partitionable.

The proof of Theorem 1.4 is now complete. $\square$

Proof of Proposition 1.5. By Proposition 1.2, it suffices to prove the cases $c(G) = |V(G)| - 2$ and $c(G) = |V(G)| - 3$. We only prove the case $c(G) = |V(G)| - 3$, since the other case is similar (and even simpler). Let G be a graph with $c(G) = c_1 + c_2$, $1 \leq c_1 \leq c_2$, and let C be a longest cycle in G with $|V(G - C)| = 3$. In view of Theorem 1.3, we may assume that $c_1 \geq 5$.

Using similar arguments as in the proofs of Claims 2.1 and 2.2, one can obtain $|N_C(G - C)| \leq c_2 + 1$. If the strict inequality holds, then, using similar arguments as those in Case 1 of the proof of Theorem 1.4, we can show that G has a (c_1, c_2)-partition. So, we may assume that $|N_C(G - C)| = c_2 + 1$. We let V_2 be any set of order c_2 satisfying $|N_C(G - C) \setminus V_2| = 1$, and let $V_1 = V \setminus V_2$. Then, from the choice of V_2, it follows that $|N_C(G - C) \cap V_1| = 1$, and this in turn implies that in $\langle V_1 \rangle$ the three vertices of $G - C$ cannot be on any cycle of length greater than 4. Since $c_1 \geq 5$, we have $c(\langle V_1 \rangle) \leq \max\{c(\langle V_1 \setminus V(G - C) \rangle), 4\} \leq |V_1 \setminus V(G - C)| = c_1$. The proposition is proved. $\square$

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